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THE FIXED-POINT PROPERTY OF $(2m-1)$-CONNECTED $4m$-MANIFOLDS

BY

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ABSTRACT. Suppose that M is a $(2m-1)$-connected smooth and compact manifold of dimension $4m$. Assume that its intersection pairing is positive definite, and denote its signature by σ. Two notions are introduced. The first is that of a (ξ, λ)-map $f: M \longrightarrow M$ where $\xi \in K(M)$ and λ an integer. It describes the concept of f preserving ξ up to multiplication by λ outside a point. The second notion is that of ξ being sufficiently asymmetric. It describes in terms of the Chern class of ξ the concept that the restrictions of ξ to the $2m$-spheres realizing a basis for $H_{2m}(M; Z)$ are sufficiently different so that no map which preserves ξ can move the spheres among themselves. One proves that (ξ, λ)-maps with ξ being sufficiently asymmetric have fixed points, except possibly when $\sigma = 2$. On taking ξ to be the complexification of the tangent bundle of M, one sees that mainfolds with sufficiently asymmetric tangent structures have the fixed point property with respect to a family of maps which includes diffeomorphisms. The question of the existence of (ξ, λ)-maps as well as the question of the preservation of the fixed-point property under products are also discussed.

1. Introduction. The problems with which this article is concerned arise in connection with the fixed-point property of $(2m-1)$-connected, smooth and compact manifolds M of dimension $4m$. Simple examples such as $S^{2m} \times S^{2m}$, and others with more complicated cohomology algebra (see §3 below) show the need to consider only manifolds with definite intersection pairing, and a restricted fixed-point property which requires that only maps $f: M \longrightarrow M$ of a certain class rather than all maps, need have fixed points. With this in mind, and with the feeling that any such theory should include, as a special case, the fixed-point theory of diffeomorphisms, or even more generally of homeomorphisms, we are lead to consider maps $f: M \longrightarrow M$ which, in a manner of speaking, preserve a given geometric structure ξ described by an element $\xi \in K(M)$. (For f to preserve ξ means only that $f^*\xi = \lambda\xi$ outside a point, with λ being an integer. Such maps will be called (ξ, λ)-maps.)

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Restricting oneself to (ξ, λ)-maps and to manifolds whose intersection pairing φ is positive definite, one sees that the Lefschetz number $L(f)$ is given by the quadratic equation $L(f) = \lambda^2 + s\lambda + 1$, with s being rational but not necessarily integral and the degree of f being λ^2. One also sees that there is a *critical threshold* θ such that $L(f) \neq 0$, for $\lambda > \theta$. For $\lambda \leqslant \theta$, the situation is more complicated, and whether a (ξ, λ)-map has a fixed point or not appears to depend on how symmetric ξ is and on the intersection pairing φ (see §3 below). Nevertheless, if ξ is *sufficiently asymmetric* in a suitable sense (given in §2), then a (ξ, λ)-map has a fixed point, except possibly when the signature σ of φ is 2. One is able to construct manifolds whose tangent structure is sufficiently asymmetric, thus obtaining manifolds which have the fixed-point property wtih respect to a class of maps which include homeomorphisms. Finally, the preservation of this property under products is considered.

An earlier version of some of the results of this paper is the announcement [5]. It should be mentioned that the notion of a (ξ, λ)-map offered here is simpler than, although equivalent to, that given in [5]. Also, the geometric structure ξ is taken to be in the Grothendieck group $K(M)$ of complex vector bundles rather than real ones, as taking ξ in $K_R(M)$ is too restrictive. Also, the assumption that the elementary divisors of $f_*: H_{2m}(M) \longrightarrow H_{2m}(M)$ are equal to λ is inadvertently left out in the statement of Theorem 3.1 as it appears in [5].

My many discussions with Ed Fadell on the topics covered in this paper helped greatly in the development of the ideas expressed herein. I take this opportunity to express my thanks to him.

2. **Statement of results.** Suppose that M^{4m} is a smooth (or PL)-manifold of dimension $4m$ and assume that M is $(2m - 1)$-connected with $m \geqslant 2$. Let $\xi \in K(M)$, where $K(M)$ is the Grothendieck group of complex vector bundles over M. A map $f: M \longrightarrow M$ is said to be a (ξ, λ)-map, λ being an integer, if, and only if, $f^* \xi | M_0 = \lambda \xi | M_0$, where $M_0 = M$-point. (Thus a diffeomorphism $f: M \longrightarrow M$ is a $(\tau^c M, 1)$-map, where $\tau^c M$ is the complexification $\tau M \otimes C$ of the tangent bundle τM of M.)

Assume next that the intersection pairing

$$\varphi: H^{2m}(M; Z) \times H^{2m}(M; Z) \longrightarrow Z$$

is *positive definite and with signature* $\sigma \neq 2$. The class of such manifolds will be denoted by M_{4m}. (They are classified in [10].) Let $c_m = c_m(\xi) \in H^{2m}(M; Z)$ be the Chern class of ξ, *and assume that* $c_m \neq 0$. Since φ is positive definite, it follows easily that the degree of a (ξ, λ)-map $f: M \longrightarrow M$ is λ^2 and that the Lefschetz number $L(f)$ is given by $L(f) = 1 + s\lambda + \lambda^2$, where s is rational and $|s| \leqslant \sigma$. (See §5 for details.) Hence $L(f) \neq 0$ for $|\lambda| > \sigma$. On the other hand, the behavior of $L(f)$ for $|\lambda| \leqslant \sigma$ is quite different, and thus σ is, in a sense, a critical threshold.

To describe the case $|\lambda| \leqslant \sigma$, let $S = \{x_1, \ldots, x_\sigma\}$ be a basis for $H_{2m}(M; Z)$, and consider the set

$$B_S = \{x \in H_{2m}(M; Z) \,|\, x^2 \leqslant \sigma^2 \leqslant \sigma^2 \mu_S\},$$

where $\mu_S = \max_i x_i^2$. Since the intersection pairing φ is positive definite, it follows that B_S is finite and, therefore, *there is a smallest positive integer* β_S *such that*

$$|a_i| < \beta_S - |\sigma|, \tag{2.1}$$

for all i, and $\Sigma_i\, a_i x_i \in B_S$.

Now let $\beta = \min\langle x, c_m\rangle$ for all $x \in H_{2m}(M; Z)$, and choose a basis $S = \{x_1, \ldots, x_\sigma\}$ such that $\langle x_1, c_m\rangle = \beta$. Observe that for a given set of integers $\{s_1, \ldots, s_\sigma\}$ such that

$$s_1 = 1 \quad \text{and} \quad s_j - s_i > 0, \ \text{ for } j > i, \tag{2.2a}$$

the change of basis $x_1 \longrightarrow x_1$ and $x_i \longrightarrow x_i - (a_i - \beta^{s_i - 1})x_1$, for $i > 1$, where $\langle x_i, c_m\rangle = a_i\beta$, allows us to assume that $S = \{x_1, \ldots, x_\sigma\}$ has been chosen so that

$$\langle x_i, c_m\rangle = \beta^{s_i}, \quad i \geqslant 1, \tag{2.2b}$$

with $\{s_i, \ldots, s_\sigma\}$ being an arbitrary set of integers satisfying (2.2a). We shall say that ξ is *sufficiently asymmetric* if, and only if, there is a basis $S = \{x_1, \ldots, x_\sigma\}$ subject to (2.2b) and such that $\beta \geqslant \beta_S$, where β_S is the integer defined by (2.1).

Note that the fact that the Chern class $c_m(\xi) \neq 0$ is implicit in the notion of asymmetry. Hence an asymmetric ξ is always nontrivial. Also note that the notion of asymmetry depends on the basis $S = \{x_1, \ldots, x_\sigma\}$. It would be interesting to find out how basis-free the notion really is.

THEOREM 2.3. *Suppose that* $M \in \mathcal{M}_{4m}$, *and let* $\xi \in K(M)$ *be sufficiently asymmetric. Then any* (ξ, λ)*-map* $f\colon M \longrightarrow M$ *has a fixed point.*

The key to the theorem is that in the critical range $|\lambda| \leqslant \sigma$ the asymmetry forces f to have a simple form homologically speaking, and the topological invariants are such that the Lefschetz number $L(f)$ will be $\neq 0$.

Now let us take $\xi = \tau^c(M)$, the complexification of the tangent bundle $\tau(M)$ of M. Note that for ξ to be sufficiently asymmetric it is necessary first that $c_m(\xi)$ be $\neq 0$, which means that dim $M = 0$ mod 8 or, equivalently, $m = 2m'$. Second, $c_m(\xi) = p_{m'}(M)$, the Pontryagin class of M, must satisfy a certain asymmetry condition. That manifolds M can be constructed with sufficiently asymmetric tangent bundles may be deduced from [10] (cf. next section). Note also that, as a consequence of Novikov's Theorem on the topological invariance of Pontryagin classes [8], we have $f^*\tau(M) = \tau(M)$ for a homeomorphism $f: M \longrightarrow M$. Thus Theorem 2.3 implies

THEOREM 2.4. *Suppose that* $M \in \mathcal{M}_{4m}$ *with m even and* $\tau^c(M) = \tau(M) \otimes C$ *sufficiently asymmetric. Then any* $(\tau^c(M), \lambda)$*-map* $f: M \longrightarrow M$ *has a fixed point. In particular, any homeomorphism of M has a fixed point.*

It is quite natural to ask if the product of two manifolds in $\mathcal{M}_{4m}$ has the fixed-point property.

THEOREM 2.5. *Suppose that* M' *and* M'' *are two manifolds in* $\mathcal{M}_{4m'}$ *and* $\mathcal{M}_{4m''}$ *respectively. Let* $\xi' \in K(M')$ *and* $\xi'' \in K(M'')$ *be sufficiently asymmetric, and set* $\xi = \xi' \boxtimes \xi''$, *where* $\boxtimes$ *is the tensor product of bundles. Then any* (ξ, λ)*-map* $f: M' \times M'' \longrightarrow M' \times M''$ *has a fixed point.*

Proofs of these theorems will be given in later sections.

REMARK 2.6. The case when $\sigma = 2$, is, of course, excluded in the results given above. However, if $\sigma = 2$ then the Lefschetz number $L(f)$ of a map $f: M \longrightarrow M$ is zero if, and only if, $\lambda = -1$. The question now arises as to whether or not one can find a manifold M of index 2 and a $(\tau^c M, * -1)$-map $f: M \longrightarrow M$ of Lefschetz number 0. A necessary and sufficient condition for this is given in Theorem 3.3 of the next section. Since this condition can be realized, it follows that the assumption that $\lambda \neq -1$ when $\sigma = 2$ is essential.

3. **Construction of (ξ, λ)-maps.** In view of the results of §2, clearly it is important to consider the question of the existence of (ξ, λ)-maps $f: M \longrightarrow M$ for a given $\xi \in K(M)$ and $\lambda \in Z$. So observe that $M_0 = M$-point is of the homotopy type of a bouquet $S^{2m} \vee \cdots \vee S^{2m}$ of $2m$-spheres, and that the homomorphism

$$\mathrm{ch}: K(M) \longrightarrow H^*(M; Q)$$

defined by the Chern character is injective since $H^*(M; Z)$ is torsion free. Hence a given map $f: M \longrightarrow M$ is a (ξ, λ)-map, for $\xi \in K(M)$ if, and only if, $f^*\mathrm{ch}_m(\xi) = \lambda\mathrm{ch}_m(\xi)$, where $\mathrm{ch}_m(\xi)$ is the mth homogeneous component of ch. This proves the first part of the following proposition.

PROPOSITION 3.1. *A given map* $f: M \longrightarrow M$ *is a* (ξ, λ)*-map if, and only if,* $f^*c_m(\xi) = \lambda c_m(\xi)$.

In either case $\lambda(\lambda - 1)\mathrm{ch}_{2m}(\xi) \in H^{4m}(M; Z) \subset H^*(M; Q)$.

PROOF. To prove the second part, let $f\colon M \longrightarrow M$ be a (ξ, λ)-map and note that there is an $\eta \in K(S^{4m})$ such that $f^*\xi = \lambda\xi + p^*\eta$, where $p\colon M \longrightarrow S^{4m}$ is a map of degree 1. Therefore

$$f^*\mathrm{ch}(\xi) = \lambda\mathrm{ch}(\xi) + p^*\mathrm{ch}(\eta),$$

which implies that

$$f^*\mathrm{ch}_{2m}(\xi) = \lambda\mathrm{ch}_{2m}(\xi) + p^*\mathrm{ch}_{2m}(\eta).$$

But $\mathrm{ch}_{2m}(\eta) \in H^{4m}(S^{4m}; Z)$. Hence $f^*\mathrm{ch}_{2m}(\xi) - \lambda\mathrm{ch}_{2m}(\xi)$ is an integral cohomology class, and to finish the proof it remains to prove that

$$f^*\mathrm{ch}_{2m}(\xi) = \lambda^2\mathrm{ch}_{2m}(\xi).$$

Now observe that

$$\begin{aligned}(\deg f)\langle [M], c_m^2\rangle &= \langle f_*[M], c_m^2\rangle \\ &= \langle [M], f^*c_m^2\rangle \\ &= \lambda^2\langle [M], c_m^2\rangle,\end{aligned}$$

since $f^*c_m = \lambda c_m$. As the intersection pairing in M is positive definite by assumption, it follows that $\deg f = \lambda^2$, which of course implies that $f^*\mathrm{ch}_{2m}(\xi) = \lambda^2\mathrm{ch}_{2m}(\xi)$.

Observe that if $\lambda = 1$, then $f^*\xi - \xi = 0$. Hence we have the following as an immediate consequence.

COROLLARY 3.2. *Suppose that $c_m(\xi) \neq 0$. Then a map $f\colon M \longrightarrow M$ is a $(\xi, 1)$-map if, and only if, $f^*\xi = \xi$. In either case f is an orientation-preserving homotopy equivalence.*

Since a (ξ, λ)-map $f\colon M \longrightarrow M$ is of degree λ^2, when $M \in \mathcal{M}_{4m}$, we are led to the question of the existence of maps $f\colon M \longrightarrow M$ of a given degree.

First we need to describe the structure of M (cf. [10]). So note that every element $x \in H_{2m}(M; Z)$ is spherical, and, since $2 \dim x = \dim M$ and $m \geqslant 2$, it can be represented by an imbedding $S^{2m} \longrightarrow M$. Following [10], we define

$$\alpha\colon H_{2m}(M; Z) \longrightarrow \pi_{2m-1}\mathrm{SO}_{2m}$$

to be the map which assigns to x the characteristic map of the normal bundle of an imbedded sphere which represents it. One can show easily (see [10]) that M is determined, up to a diffeomorphism, by α and the intersection pairing φ. To compute α, consider the map $S \oplus HJ\colon \pi_{2m-1}\mathrm{SO}_{2m} \longrightarrow \pi_{2m-1}\mathrm{SO} \oplus Z$, where S is the homomorphism induced by the natural injection $\mathrm{SO}_{2m} \longrightarrow \mathrm{SO}$, J being the J-homomorphism $\pi_{2m-1}\mathrm{SO}_{2m} \longrightarrow \pi_{4m-1}S^{2m}$ and H being the Hopf-invariant $\pi_{2m-1}S^{2m} \longrightarrow Z$. Recall that $S \oplus HJ$ is injective [6] and that

$$\pi_{2m-1}\mathrm{SO} = \begin{cases} Z, & m \text{ even}, \\ Z_2 \text{ or } 0, & m \text{ odd}. \end{cases}$$

Thus the map $S\alpha$: $H_{2m}(M; Z) \longrightarrow \pi_{2m-1}\mathrm{SO}$ defines naturally an element in $H^{2m}(M; Z)$ or $H^{2m}(M; Z_2)$ according to the parity of m. Moreover, when $m = 2m'$, $S\alpha$ is, up to a multiple, just the Pontryagin class $p_{m'}$ of M. *Now, by definition, let* $\chi \in H^{2m}(M; Z)$ *be an element such that* $\chi = S\alpha$, *when* m *is even, and* $\chi = S\alpha \bmod 2$ *otherwise.*

Let us now return to the problem of the existence of maps f: $M \longrightarrow M$ with a given degree. We shall consider here a special case only. This is sufficient for the purposes of this paper, and the general problem is still elusive. So let γ: $H^{2m}(M; Z) \longrightarrow H^{2m}(M; Z)$ be a monomorphism such that

$$\varphi(\gamma x, \gamma y) = \lambda^2 \varphi(x, y) \tag{3.3a}$$

for all $x, y \in H^{2m}(M; Z)$, where λ is a given integer and φ is the intersection pairing in M. Next choose bases $\{x_1, \ldots, x_\sigma\}$ and $\{x'_1, \ldots, x'_\sigma\}$ for $H^{2m}(M; Z)$ such that

$$\gamma x_i = \lambda_i x'_i, \tag{3.3b}$$

where $\lambda_i \in Z$ with $\lambda_1 | \lambda_2 | \cdots | \lambda_\sigma$. (The integers λ_i are the elementary divisors of γ.) Now with these conditions we have the following theorem (cf. [11, Lemma 10] and [10, Theorem 5]).

THEOREM 3.4. *Suppose that* $\gamma(\chi) = \lambda\chi$. *Then there is a map* f: $M \longrightarrow M$ *such that* γ *is the induced homomorphism on cohomology if, and only if* $J(\lambda_i(\lambda - 1)\chi(y'_i)) = 0$, *for* $i = 1, \ldots, \sigma$, *where* $y'_i = [M] \cap x'_i$ *and* J *is the* J-*homomorphsim of* $\pi_{2m-1}\mathrm{SO}$ *to the stable stem* $\Pi_{2m-1}S$.

The condition $J(\lambda_i(\lambda - 1)\chi(y_i)) = 0$ given in the theorem is equivalent to the condition $J(\lambda_i(\lambda - 1)(S\alpha)(y_i)) = 0$, for $i = 1, \ldots, \sigma$, because $\chi = S\alpha$, when m is even, and $\chi = S\alpha \bmod 2$ when m is odd (in which case the image of J is of order 2 or 0; see [1]). Thus the condition in the theorem is a statement on the stable spherical homotopy type of the tangent bundle. This can be seen clearly in the case presented by the following corollary. (See [5].)

COROLLARY 3.5. *With the same conditions as those of Theorem* 3.4, *assume that* $\lambda = \lambda_i$, *for all* i. *Then there is a map* f: $M \longrightarrow M$ *such that* f^*: $H^{2m}(M; Z) \longrightarrow H^{2m}(M; Z)$ *is* γ *if, and only if,* $J(\lambda(\lambda - 1)S\alpha(x)) = 0$, *for all* $x \in H_{2m}(M; Z)$.

Note that $\lambda(\lambda - 1) = 0 \bmod 2$, and therefore the conditions of Corollary 3.5 are always satisfied when m is odd. Moreover, $\lambda(\lambda - 1) = 2$ when $\lambda = -1$

and maps $f\colon M \longrightarrow M$ such that $f^* S\alpha = -S\alpha$ exist only if the tangent bundle of M is is suitably restricted. This is the restriction alluded to in §2 of the discussion of the condition that the signature $\sigma \neq 2$ is required in the main results.

The proof of Theorem 3.4 depends in an essential way on the description of M given in [10]. Let $\{h_i\}$, $\{h_i'\} \subset H_{2m}(M; Z)$ be the bases, dual to the bases $\{x_i\}$ and $\{x_i'\}$ given above, defined by the equations

$$\langle h_i, x_j\rangle = \delta_{ij} = \langle h_i', x_j'\rangle,$$

for $i, j = 1, \dots, \sigma$. According to [10], $\{h_i\}$ and $\{h_i'\}$ provide two cell decompositions of M,

$$(3.6) \qquad \begin{aligned} M &= (S_1^{2m} \vee \cdots \vee S_\sigma^{2m}) \cup_\beta D^{4m}, \\ M' &= (S_1'^{2m} \vee \cdots \vee S_\sigma'^{2m}) \cup_{\beta'} D^{4m}, \end{aligned}$$

where S_i^{2m} and $S_i'^{2m}$ represent $\{h_i\}$ and $\{h_i'\}$ respectively. Moreover,

$$(3.7) \qquad \begin{aligned} \beta &= \sum_i h_i \circ \beta_i + \sum_{i<j} a_{ij}[h_i, h_j], \\ \beta' &= \sum_i h_i \circ \beta_i' + \sum_{i<j} a_{ij}'[h_i', h_j'] \end{aligned}$$

with the sum corresponding to the canonical decompositions

$$\pi_{4m-1}(S_1^{2m} \vee \cdots \vee S_\sigma^{2m}) = \sum_i \pi_{4m-1} S_i^{2m} + \sum_{i<j} \pi_{4m-1} S_{ij}^{4m-1}$$

familiar in homotopy theory. We also have

$$(3.8) \qquad \begin{cases} a_{ij} = \varphi(x_i, x_j), \\ \beta_i = x_i \circ \beta = J(\alpha(y_i)), \end{cases} \quad \text{and} \quad \begin{cases} a_{ij}' = \varphi(x_i', x_j'), \\ \beta_i' = x_i' \circ \beta' = J(\alpha(y_i')) \end{cases}$$

with $y_i = [M] \cap x_i$, $y_i' = [M] \cap x_i'$, where we regard x_i, x_i' as maps of $M_0 = M - D^{4m}$ and M_0' to S^{4m} inducing x_i and x_i' respectively.

Now it is quite easy to find a map $f_0\colon M_0' \longrightarrow M_0$ such that the induced homomorphism $f_0^*\colon H^{2m}(M_0; Z) \longrightarrow H^{2m}(M_0'; Z)$ is the given map γ. Note that the adjoint $\gamma_\#\colon H_{2m}(M'; Z) \longrightarrow H_{2m}(M; Z)$ takes h_i' to $\lambda_i h_i$. The CW-representation of M given in (3.6) shows that f_0 can be extended to a map $f\colon M' \longrightarrow M$ if, and only if, $f_0\beta$ and $\lambda^2\beta'$ are homotopic to each other. Because of the special forms (3.7) of β and β', it follows that $f_0\beta$ is homotopic to $\lambda^2\beta'$ if, and only if, the map f_0 is such that

$$(3.9) \qquad a_{ij}'[h_i', h_j'] \xrightarrow{f_0} \lambda^2 a_{ij}[h_i, h_j], \quad h_i' \circ \beta_i' \xrightarrow{f_0} \lambda^2 h_i \circ \beta_i.$$

Now, since $h_i' \xrightarrow{\gamma_\#} \lambda_i h_i$, we have

$$a_{ij}'[h_i', h_j'] \longrightarrow \lambda_i \lambda_j a_{ij}'[h_i, h_j]$$

and

$$\lambda^2 a_{ij} = \varphi(\gamma x_i, \gamma x_j) = \varphi(\lambda_i x_i', \lambda_j x_j') = \lambda_i \lambda_j a_{ij}',$$

according to (3.8). Hence,

$$a_{ij}'[h_i', h_j'] \longrightarrow \lambda^2 a_{ij}[h_i, h_j],$$

the first statement of (3.9).

To prove the second assertion, we consider first the commutative diagram

$$\begin{array}{ccccccc}
 & \cdots \longrightarrow \pi_{2m} S^{2m} & \xrightarrow{\partial} & \pi_{2m-1}\mathrm{SO}_{2m} & \xrightarrow{S} & \pi_{2m-1}\mathrm{SO}_{2m+1} \longrightarrow 0 & \\
 & & & \downarrow J & & \downarrow J & \\
0 \longrightarrow & \pi_{4m}(\Omega S^{2m+1}, S^{2m}) & \xrightarrow{\partial} & \pi_{4m-1} S^{2m} & \xrightarrow{\Sigma} & \pi_{4m} S^{2m+1} \longrightarrow 0 & \\
(3.10) & & & \cong \downarrow & & & \\
 & & & \pi_{4m-2}\Omega S^{2m} & \xrightarrow{H} & \pi_{4m-2}(\Omega S^{2m}, S^{2m-1}) \longrightarrow \cdots & \\
 & & & & & \cong & \\
 & & & & & Z &
\end{array}$$

where S and Σ are the appropriate suspension homomorphisms, while J and H are the J-homomorphism and the Hopf-invariant. Also, the horizontal lines are those defined by the exact sequences of the fibrations $\mathrm{SO}_{2m} \longrightarrow \mathrm{SO}_{2m+1} \longrightarrow S^{2m}$ and of the pair $(\Omega S^{2m+1}, S^{2m})$. Hence, to prove the second assertion of (3.9) that $h_i' \circ \beta_i' \xrightarrow{f_0} \lambda^2 h_i \circ \beta_i$ is equivalent to showing that $\lambda^2 h_i \circ \beta_i$ and $(\lambda_i h_i) \circ \beta_i' = f_0(h_i' \circ \beta_i')$ have the same Hopf-invariant and the same suspension. Now the well-known distributive laws in homotopy theory [4] show that

$$(\lambda_i h_i) \circ \beta_i' = h_i \circ (\lambda_i \beta_i' + \tfrac{1}{2}\lambda_i(\lambda_i - 1)\, [\iota, \iota]\, H(\beta_i')),$$

where $H(\beta_i')$ is the Hopf-invariant of β_i' and $[\iota, \iota]$ is the Whitehead product of the identity $\iota \in \pi_{2m} S^{2m}$. Therefore it suffices to show that $(\lambda_i \beta_i' + \tfrac{1}{2}\lambda_i(\lambda_i - 1)[\iota, \iota]H(\beta_i'))$ and $\lambda^2 \beta_i$ have the same Hopf-invariant and the same suspension. Since $H([\iota, \iota]) = 2$, we see immediately that

$$H(\lambda_i \beta_i' + \tfrac{1}{2}\lambda_i(\lambda_i - 1)\, [\iota, \iota]\, H(\beta_i')) = \lambda_i^2\, H(\beta_i'). \tag{3.11}$$

Also we have, over the rationals,

$$\gamma_\#(y_i') = \gamma_\#([M] \cap \gamma(x_i/\lambda_i)) = \gamma_\#[M] \cap (x_i/\lambda_i) = (\lambda^2/\lambda_i) y_i, \tag{3.12}$$

and, therefore, over the integers,

$$\lambda_i(\gamma_\# y_i') = \lambda^2(y_i) \tag{3.12a}$$

implying that

$$\lambda_i^2 \lambda^2 y_i'^2 = \lambda_i^2 (\gamma_\# y_i')^2 = \lambda^4 y_i^2, \quad \text{or} \quad \lambda_i^2 y_i'^2 = \lambda^2 y_i^2,$$

for $i = 1, \ldots, \sigma$, where y^2 is the self-intersection of y with itself. But, according to [10], $H(\beta_i) = HJ\alpha(y_i) = y_i^2$. Hence equation (3.11) implies the equations

$$H(\lambda_i \beta_i' + \tfrac{1}{2}\lambda_i(\lambda_i - 1)[\iota, \iota]\, H(\beta_i')) = H(\lambda^2 \beta_i),$$

for $i = 1, \ldots, \sigma$.

Finally we need to prove that

$$\Sigma\, (\lambda_i \beta_i' + \tfrac{1}{2}\lambda_i(\lambda_i - 1)[\iota, \iota]\, H(\beta_i')) = \Sigma\, (\lambda^2 \beta_i), \tag{3.13}$$

Σ being the suspension homomorphisms. Now the left-hand side of (3.16) is equal to

$$\lambda_i \Sigma\, \beta_i' = \lambda_i \Sigma\, J\alpha(y_i') = \lambda_i J(S\alpha)\,(y_i'),$$

according to diagram (3.10). Similarly,

$$\Sigma\, (\lambda^2 \beta_i) = \lambda^2 \Sigma\, J\alpha(y_i) = \lambda^2 J(S\alpha)\,(y_i).$$

But (3.12a) implies that

$$\lambda^2 (S\alpha)\,(y_i) = (S\alpha)\,(\gamma_\#(\lambda_i y_i')),$$

and therefore

$$\lambda^2 (S\alpha)(y_i) - \lambda_i (S\alpha)(y_i') = (S\alpha)(\gamma_\#(\lambda_i y_i')) - \lambda_i (S\alpha)(y_i').$$

Now, by assumption, we have $(S\alpha)(\gamma_\# y) = \lambda(S\alpha)(y)$, for all $y \in H_{2m}(M; Z)$, when m is even, and $(S\alpha)(\gamma_\# y) = \lambda(S\alpha)(y)$ mod 2 when m is odd. Since

$$\text{image}(\pi_{2m-1}\text{SO} \xrightarrow{J} \pi_{2m-1}S)$$

is of order 2 or 0, when m is odd, we conclude that

$$J(\lambda^2 (S\alpha)(y_i) - \lambda_i (S\alpha)(y_i')) = J(\lambda\lambda_i S\alpha(y_i') - \lambda_i (S\alpha)(y_i')) = 0,$$

for $i = 1, \ldots, \sigma$. This proves (3.13) and, hence, Theorem 3.4.

The following special case of Theorem 3.4 is noteworthy. Suppose that $\lambda = 1$ and that $\gamma: H^{2m}(M; Z) \longrightarrow H^{2m}(M; Z)$ preserves the stable tangential structure χ. According to Theorem 3.4, we are able to realize γ by a map $f: M \longrightarrow M$ such that $f^*\tau M$ is stably equivalent to τM; i.e., f is tangential. Hence $f \mid M_0$, with $M_0 = M - 4m$-disk, is homotopic to an immersion $M_0 \longrightarrow M_0$ which is regularly homotopic to an imbedding, since $3(2m + 1) < 2(2m)$. This imbedding is in turn diffeotopic to a diffeomorphism $f_0: (M_0, \partial M_0) \longrightarrow (M_0, \partial M_0)$, since $4m = \dim M_0 > 5$. Thus we obtain the following

COROLLARY 3.14. *Given a quadratic automorphism* $\gamma: H^{2m}(M; Z) \longrightarrow$

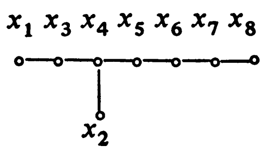
$H^{2m}(M; Z)$ such that $\gamma(\chi) = \chi$, there is a homeomorphism $f\colon M \longrightarrow M$ such that $f^* = \gamma$. Moreover, f is a diffeomorphism on M-{point}.

4. Examples, applications, and remarks. Let us consider now the problem of characterizing those manifolds $M \in \mathcal{M}_{4m}$ which have the fixed-point property with respect to maps $f\colon M \longrightarrow M$ which preserve the tangent structure in the sense of §2. According to Theorem 2.4 of §2, if the $\tau^c(M) = \tau(M) \otimes C$ is sufficiently asymmetric, then any map $f\colon M \longrightarrow M$ which preserves $\tau^c(M)$ has a fixed point. So the question arises as to whether one can allow a greater degree of symmetry in $\tau^c(M)$ and still retain the fixed-point property. We shall see that the answer depends on the intersection pairing φ and its group of automorphisms.

Consider first the case when the tangent structure is trivial and hence is fully symmetric.

PROPOSITION 4.1. *Suppose that $M \in \mathcal{M}_{4m}$, $m > 4$, is almost-parallelizable and that the intersection pairing φ has signature* 8. *Then there is a homotopy equivalence $f\colon M \longrightarrow M$ which is fixed-point free.*

PROOF. Recall, that, by the definition of the class $\mathcal{M}_{4m}$, φ is positive definite. Also, $m > 4$ implies that φ is of type II, and therefore φ is equivalent to the form associated with the Exceptional Lie group E_8. Precisely, there is a basis $\{x_1, \ldots, x_8\}$ for $H_{2m}(M; Z)$ such that the matrix $[\varphi_\#(x_i, x_j)]$, $\varphi_\#$ being the adjoint of φ, is the Cartan matrix of the Exceptional Lie group E_8 whose Dynkin diagram is



(see [2]). Denote by ρ_i the reflection defined by x_i for $i = 1, \ldots, 8$. Then $\gamma = \rho_8\rho_7\rho_6\rho_5\rho_2\rho_3\rho_1$ is a quadratic automorphism of $H_{2m}(M; Z)$ of trace -2. Now the conditions of Corollary 3.5 are satisfied, and therefore γ can be realized by a map $f\colon M \longrightarrow M$ of degree 1. Since $\mathrm{tr}(f^*) = -2$, the Lefschetz number $L(f)$ is 0, and therefore f is homotopic to a fixed-point free map [3].

REMARK 4.2. The map f constructed above can be taken to be a homeomorphism possibly with fixed points, but I do not know whether its fixed points can be cancelled against each other while keeping f a homeomorphism throughout.

REMARK 4.3. The conditions on M can be relaxed somewhat: all one needs are enough restrictions to ensure the realizability of γ. For example, this is so if m is odd.

In the course of proving Theorem 2.1 of §2 in the next section, we shall see that the high degree of asymmetry imposed on the geometric structure ξ

makes those maps which respect it homologically simple. One can achieve the same effect by specializing the intersection pairing φ but allowing a greater degree of symmetry in ξ. The following illustrates what can be done in this vein.

PROPOSITION 4.4. *Suppose that M is in $\mathcal{M}_{8m}$, and assume that the intersection pairing φ is equivalent to that associated with the Exceptional Lie group E_8. Let $\{x_i\}$ be a basis for $H_{4m}(M; Z)$ defined by the Dynkin diagram of E_8, and assume that the Pontryagin numbers $p_m(x_i)$ are mutually distinct and positive. Then any homeomorphism $f\colon M \longrightarrow M$ has a fixed point.*

PROOF. Assume that the basis is indexed in such a way that $0 < \langle x_1, p_m\rangle < \langle x_2, p_m\rangle < \cdots < \langle x_8, p_m\rangle$. Observe next that the induced homomorphism $f_*\colon H_{4m}(M; Z) \longrightarrow H_{4m}(M; Z)$ is an element of the Weyl group of E_8 and, therefore, that $f_*(x_1) = \Sigma_i\, a_{1i}x_i$, with all coefficients a_{1i} having the same sign. Now we evaluate the Pontryagin class on both sides of the equation to get

$$\langle f_*(x_1), p_m\rangle = \sum_i a_{1i}\langle x_i, p_m\rangle.$$

But

$$\langle f_*(x_1), p_m\rangle = \langle x_1, f^*p_m\rangle = \langle x_1, p_m\rangle,$$

since f is a homeomorphism [8]. Hence

$$\langle x_1, p_m\rangle = \sum_i a_{1i}\langle x_i, p_m\rangle.$$

But the coefficients a_{1i} have the same sign, and $\langle x_1, p_m\rangle < \langle x_i, p_m\rangle$ for all $i > 1$. Therefore $a_{11} = +1, a_{1i} = 0, i > 1$.

Consider next $f_*x_2 = \Sigma_i\, a_{2i}x_i$ with the coefficients all having the same sign. Evaluating p_m on both sides of the equation, we get

$$\langle x_2, p_m\rangle = \sum_i a_{2i}\langle x_2, p_m\rangle.$$

This implies immediately that $a_{2i} = 0$, for $i \geqslant 3$, because of the condition on the magnitudes of the values of the Pontryagin class. To show that $a_{22} \neq 0$, assume otherwise; then $f_*x_2 = a_{21}x_1$, implying that $f_*(x_2 - a_{21}x_1) = 0$. But f_* is an isomorphism, and hence the assumption that $a_{22} = 0$ leads to a contradiction. Again the condition that $0 < \langle x_1, p_m\rangle < \langle x_2, p_m\rangle$ implies that $a_{21} = 0$ and $a_{22} = 1$. The rest of the argument follows similarly.

One can generalize Proposition 4.4 to those (τ^cM, λ)-maps $f\colon M \longrightarrow M$ with the property that the elementary divisors of $f_*\colon H_{4m}(M; Z) \longrightarrow H_{4m}(M; Z)$ are all equal.

The key fact in the proof of Proposition 4.4 is that the induced homomorphism f_* is an element of the Weyl group. Therefore it appears possible that one might be able to prove results of this kind for manifolds whose intersection pairings are characterized by root systems of Lie groups, e.g., when the signature is 24 (cf. [7]).

5. **Proof of Theorem 2.3.** In fact we will prove a slightly stronger theorem. Suppose that $M \in \mathcal{M}_{4m}$, and note that to prove that a given map $f: M \longrightarrow M$ has a fixed point it suffices to prove that the Lefschetz number $L(f)$ of f is $\neq 0$. So first note that

$$\varphi(f^*x, f^*y) = \langle [M], (f^*x)(f^*y)\rangle = \langle f_*[M], xy\rangle$$

$$= (\deg f)\,\langle [M], xy\rangle = (\deg f)\varphi(x,y),$$

for all $x, y \in H^{2m}(M; Z)$. Hence if $\deg f = 0$, then $f^*x = 0$, for all $x \in H^{2m}(M; Z)$, since φ is positive definite. This implies that $L(f) = 1$ in that case. So let us assume hereon that $\deg f \neq 0$.

PROPOSITION 5.1. *Assume that there is a nonzero element $c \in H^{2m}(M; Z)$ such that $f^*c = \lambda c$, for some $\lambda \in Z$. Then the induced homomorphism f_C^* : $H^{2m}(M; C) \longrightarrow H^{2m}(M; C)$ is diagonalizable with all the characteristic values equal to $|\lambda|$ in absolute value.*

PROOF. Let $\mu \in C$ be a characteristic value for f_c^*, and let $x \in H^{2m}(M; Z)$ be the corresponding characteristic vector. Then

$$\varphi_C(f^*x, f^*n) = \mu\bar{\mu}\varphi_C(x, x) = (\deg f)\varphi_C(x, x),$$

where φ_C is the Hermitian form induced by φ. Hence $\deg f = |\mu|^2$, since φ is positive definite. In particular, we have $\deg f = \lambda^2$ because λ itself is a characteristic value for f^* with characteristic vector c. This proves the proposition.

Now, the fact that $\deg f = \lambda^2$ implies that the Lefschetz number $L(f)$ of f is given by the equation $L(f) = 1 + s\lambda + \lambda^2$, where s is a real number. But $L(f)$ itself is integral. Hence s is in fact rational.

Next we prove that $s = \lambda\sigma$, where σ is the signature of φ, provided a certain condition is satisfied. (See the definition of the notion of sufficient asymmetry of $\xi \in K(M)$ in §2.)

CONDITION 5.2. *For a given $c \in H^{2m}(M; Z)$, there is a basis $S = \{x_i\}$ for $H^{2m}(M; Z)$ such that $\langle x_i, c\rangle = \beta^{s_i}$, with $s_1 = 1$ and $s_j - s_i > 0$, for all $j > i$, with $\beta \geqslant \beta_S$ where β_S is the bound defined by S in §2.*

Observe that if ξ is sufficiently asymmetric, then the Chern class $c = c_m(\xi) \in H^{2m}(M; Z)$ satisfies Condition 5.2.

Now Theorem 2.3 follows easily from the following

THEOREM 5.3. *Suppose that $f: M \longrightarrow M$ is a map such that $f^*c = \lambda c$, for some $\lambda \in H^{2m}(M; Z)$. Assume that c satisfies Condition 5.2 and that $|\lambda| \leqslant \sigma$, with σ being the signature of M. Then the Lefschetz number $L(f)$ of f is* $1 +$

$\sigma\lambda + \lambda^2$. *In particular, f has a fixed point if either* $\sigma \neq 2$ *or* $\lambda \neq -1$.

To prove Theorem 5.3 it suffices to prove the following.

LEMMA 5.4. *Suppose that* $|\lambda| \leqslant \sigma$. *Then* $f_*(x_i) = \lambda x_i$, *for all i where* $S = \{x_i\}$ *is the basis in Condition* 5.2.

PROOF. Let us write $f_*(x_i) = \Sigma_j a_{ij}x_j$. Now, on evaluating c on both sides of the equation and noting that

$$\langle f_* x_i, c\rangle = \langle x_i, f^*c\rangle = \lambda\langle x_i, c\rangle,$$

we obtain the equation

$$(5.5)_1 \qquad \lambda\beta^{s_i} = a_{i1}\beta + \sum_{j\geqslant 2} a_{ij}\beta^{s_j}.$$

This clearly implies that $a_{i1} = 0 \bmod \beta$, since $s_i \geqslant 2$, for $i \geqslant 2$. But

$$|\varphi'(f_*x_i, f_*x_i)| = \lambda^2 |\varphi'(x_i, x_i)| \leqslant \lambda^2\mu_S \leqslant \sigma^2\mu_S$$

where φ' is the adjoint of the intersection pairing φ and $\mu_S = \max_i \varphi'(x_i, x_i)$. Hence $f_*x_i \in B_S$, and therefore

$$|a_{i1}| < \beta_S - |\sigma| < \beta_S \leqslant \beta.$$

This implies that $a_{i1} = 0$, and thus equation $(5.5)_1$ can be reduced to

$$(5.5)_2 \qquad \lambda\beta^{s_i} = \beta^{s_2}\left(a_{i2} + \sum_{j\geqslant 3} a_{ij}\beta^{s_j - s_2}\right),$$

with $s_j - s_2 > 0$. Repeating this process enough times, one shows that $a_{ij} = 0$, for all $j \leqslant i-1$, thereby reducing $(5.5)_2$ to the equation

$$(5.5)_3 \qquad \lambda\beta^{s_i} = \beta^{s_i}\left(a_{ii} + \sum_{j> i} a_{ij}\beta^{s_j - s_i}\right),$$

with $s_j - s_i > 0$, for $j > i$. Now divide by β^{s_i} and reduce modulo β to obtain the congruence $\lambda - a_{ii} = 0 \bmod \beta$. But

$$|\lambda - a_{ii}| \leqslant |\lambda| + |a_{ii}| \leqslant |\lambda| + \beta_S - |\sigma| < \beta$$

and therefore $\lambda = a_{ii}$.

To finish the proof of the lemma, one needs to prove that $a_{ij} = 0$ for $j > i$. This can be done in the same manner as in the first part of the proof.

6. **Proof of Theorem 2.5.** Again we shall prove a slightly stronger theorem. So let M' and M'' be in $\mathcal{M}_{4m'}$ and $\mathcal{M}_{4m''}$ with signatures σ' and σ'' respectively, and suppose $c' \in H^{2m'}(M'; Z)$ and $c'' \in H^{2m''}(M''; Z)$ satisfy Condition 5.2 of §5.

THEOREM 6.1. *Suppose that* $f\colon M' \times M'' \longrightarrow M' \times M''$ *is a map such that*

$f^*(c' + c'') = \lambda(c' + c'')$ where $\lambda \in Z$. Then f has a fixed point.

If one specializes c' and c'' to be the Chern classes $c_m(\xi')$ and $c_m(\xi'')$ of sufficiently asymmetric ξ' and ξ'', then Condition 5.2 is satisfied and Theorem 2.5 becomes an immediate consequence of Theorem 6.1.

We shall prove Theorem 6.1 by treating different cases of the problem separately. By definition, let $f'\colon M' \longrightarrow M'$ and $f''\colon M'' \longrightarrow M''$ be the composites

$$M' \xrightarrow{i'} M' \times M'' \xrightarrow{f} M' \times M'' \xrightarrow{p'} M',$$
$$M'' \xrightarrow{i''} M' \times M'' \xrightarrow{f} M' \times M'' \xrightarrow{p''} M'',$$

where i', i'' are the natural injections while p', p'' are the natural projections.

The following proposition resembles one given in [3].

PROPOSITION 6.2. *Suppose that $m' \neq m''$. Then $L(f) = L(f')L(f'')$, and therefore f has a fixed point.*

PROOF. By the Künneth Formula we have

$$H^*(M', Z) \otimes H^*(M; Z) \cong H^*(M' \times M''; Z).$$

Since $m' \neq m''$, it follows that $f^* = f'^* \otimes f''^*$. Hence $L(f) = L(f') \cdot L(f'')$, because the homomorphisms $f'^* \otimes 1$ and $1 \otimes f''^*$ commute with each other and are both diagonalizable. To prove that f has a fixed point (or $L(f) \neq 0$), not that the equation $f^*c = \lambda c$, with $c = c' + c''$, implies that $f'^*(c') = \lambda c'$ and $f''^*(c'') = \lambda c''$. But c' and c'' satisfy Condition 5.2 of §5. Hence $L(f) \neq 0$ as required.

Next we shall consider the case when, say, $m' = m'' = m$. It is convenient to let $M = M' \times M''$. The following simple lemma is the key to the analysis.

LEMMA 6.3. *Let $x = x' + x''$ be in $H^{2m}(M'; Z)$, with x', x'' being nonzero elements of $H^{2m}(M'; Z)$ and $H^{2m}(M''; Z)$ respectively. Then $x^4 \neq 0$, and hence if $y \in H^{2m}(M'; Z)$ or $H^{2m}(M''; Z)$, then $f^*(y) \in H^{2m}(M'; Z)$ or $H^{2m}(M''; Z)$.*

PROOF. On expanding x^4 we obtain

$$x^4 = x'^4 + 4x'^3x'' + 6x'^2x''^2 + 4x'x''^3 + x''^4.$$

But $x'^3 = 0$, $x''^3 = 0$ and $x'^2x''^2$ is a nonzero multiple of the generator of $H^{8m}(M; Z)$. Hence $x^4 \neq 0$. To prove the second assertion, just note that $y^2 = 0$. The lemma is thereby proved.

Thus the homomorphism $f^*\colon H^{2m}(M; Z) \longrightarrow H^{2m}(M; Z)$ can be described in one of the following ways.

Case I. *f^* preserves factors; i.e.,*

$$f^*H^{2m}(M'; Z) \subset H^{2m}(M'; Z) \quad \text{and} \quad f^*H^{2m}(M''; Z) \subset H^{2m}(M''; Z).$$

Case II. *f^* takes both factors in one or the other factor*; *i.e.*,

$$f^*H^{2m}(M;Z) \subset H^{2m}(M';Z) \quad \text{or} \quad f^*H^{2m}(M;Z) \subset H^{2m}(M'';Z).$$

Case III. *f^* interchanges the two factors*; *i.e.*,

$$f^*H^{2m}(M';Z) \subset H^{2m}(M'';Z), \quad f^*H^{2m}(M'';Z) \subset H^{2m}(M';Z).$$

Cases I and II are easier than Case III and are dealt with by the following

PROPOSITION 6.4. *Assume that either Case* I *or Case* II *holds. Then* $L(f) = L(f') \cdot L(f'')$ *and hence* $L(f) \neq 0$.

The proof is similar to that for Proposition 6.2 and, therefore, will not be given here.

Now we come to Case III.

PROPOSITION 6.5. *Suppose that Case* III *holds. If* $\lambda = 0$, *then* $L(f) = 1$. *If* $\lambda \neq 0$, *then* $\sigma' = \sigma = \sigma''$ *and* $L(f) = 1 + s\lambda^2 + \lambda^4$, *with s rational and* $|s| \leqslant \sigma$. *Moreover,* $|s| = \sigma$ *when* $\lambda^2 \leqslant \sigma$. *In either case* $L(f) \neq 0$ *and therefore f has a fixed point.*

PROOF. First observe that $\mathrm{tr}(f_*)$ is 0 in dimensions $2m$ and $6m$, and hence $L(f) = 1 + s\lambda^2 + \lambda^4$, with s rational. Now observe that

$$f^*c = f^*c' + f^*c'' = \lambda c' + \lambda c'',$$

and hence that

$$f^*c' = \lambda c'', \qquad f^*c'' = \lambda c'.$$

Therefore

$$f^*(c'^2 + c''^2) = \lambda^2(c'^2 + c''^2),$$

implying that

$$f^*(\mu' + \mu'') = \lambda^2(\mu' + \mu''),$$

where $\mu' \in H^{2m}(M';Z)$ and $\mu'' \in H^{2m}(M'';Z)$ are the generators given by the orientations. Now, by definition, let

$$\gamma\colon H^{2m}(M;Z) \times H^{2m}(M;Z) \longrightarrow Z$$

be the pairing defined by $(x, y) \longrightarrow \varphi'(x', y') + \varphi''(x'', y'')$ where $x = x' + x''$ and $y = y' + y''$ with $x', y' \in H^{2m}(M';Z)$ and $x'', y'' \in H^{2m}(M'';Z)$. (In fact, γ is the pairing defined by the connected sum $M' \# M'' \subset M$.) It is easy to verify that $\gamma(f^*x, f^*y) = \lambda^2\gamma(x, y)$, for all $x, y \in H^{2m}(M;Z)$. Note also that γ is positive definite. Hence all the characteristic values of $f^*\colon H^{2m}(M;C) \longrightarrow$

$H^{2m}(M; C)$ are equal to $|\lambda|$ in absolute value. Thus, if $\lambda = 0$, it follows that $f^* = 0$ and, hence $L(f) = 1$, thereby proving the first part of the proposition.

Let us consider the next case when $\lambda \neq 0$. But this implies that the homomorphism $f^*: H^{2m}(M; Z) \longrightarrow H^{2m}(M; Z)$ is injective, since γ is a positive definite form, and hence $\sigma' = \sigma'' = \sigma$.

Consider next the map $g = f^2: M \longrightarrow M$ obtained by squaring f. Clearly the induced homomorphism

$$g^* = (f^2)^* = (f^*)^2: H^*(M; Z) \longrightarrow H^*(M; Z)$$

takes $H^*(M'; Z)$ to $H^*(M'; Z)$ and $H^*(M''; Z)$ to $H^*(M''; Z)$. In fact g^* on $H^*(M'; Z)$ and $H^*(M''; Z)$ is induced by the composites

$$M' \xrightarrow{i'} M' \times M'' \xrightarrow{g} M' \times M'' \xrightarrow{p'} M'$$

and

$$M'' \xrightarrow{i''} M' \times M'' \xrightarrow{g} M' \times M'' \xrightarrow{p''} M''$$

respectively. Moreover

$$g^*(c') = \lambda^2 c' \quad \text{and} \quad g^*(c'') = \lambda^2 c'',$$

and therefore, according to Proposition 5.1.of §5, the induced homomorphism on cohomology with complex coeficients is diagonalizable. Hence we can find a basis $S = \{x_i'\}$, for $H^{2m}(M'; C)$, such that $g^*(x_i') = \zeta_i^2 x_i'$, for all i, with $\zeta_i \in C$ being of absolute value $|\lambda|$. This implies that there is a basis $S'' = \{x_i''\}$ for $H^{2m}(M''; C)$ such that $f^*x_i' = \zeta_i x_i''$, $f^*x_i'' = \zeta_i x_i'$ for all i. Now note that $x_i' \otimes x_j''$ is a basis for $H^{2m}(M'; C) \otimes H^{2m}(M''; C) \subset H^{4m}(M; C)$ and that $x_i' \otimes x_j'' \xrightarrow{f^*} \zeta_i \zeta_j x_j' \otimes x_i''$, for all $i, j = 1, \ldots$. Therefore we have

$$\operatorname{tr} f^{2m} = 0, \quad \operatorname{tr} f^{4m} = s\lambda^2, \quad \operatorname{tr} f^{6m} = 0, \quad \text{and} \quad \operatorname{tr} f^{8m} = \lambda^4,$$

where s is a rational number such that $|s| \leqslant \sigma$ and f^i is the homomorphism $H^i(M; C) \longrightarrow H^i(M; C)$ induced by f. Hence $L(f) = 1 + s\lambda^2 + \lambda^4$. Now the fact that $|s| \leqslant \sigma$ implies that $L(f) \neq 0$, for $\lambda^2 > \sigma$. If $\lambda^2 \leqslant \sigma$, consider the map $g = f^2: M \longrightarrow M$ again and note that by applying Lemma 5.4 to each of the two factors M' and M'' one may deduce that the induced homomorphism $g^*: H^{2m}(M; Z) \longrightarrow H^{2m}(M; Z)$ is just multiplication by λ^2. Finally, arguing as above, we find that $\operatorname{tr} f^{4m} = \sigma\lambda^2$, implying that $L(f) = 1 + \sigma\lambda^2 + \lambda^4$, for $\lambda^2 \leqslant \sigma$.

This proves Proposition 6.5, and, hence, Theorem 6.1.

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